Generalized Quasi-Linearization Method

D. K. SCHARMACK*
Honeywell, Inc., Minneapolis, Minn.
AND
N. R. SANDELL JR.†
MIT, Cambridge, Mass.

A new algorithm based on initial value techniques for numerical solution of two-point boundary value problems is presented. The initial guess consists of a starting trajectory that does not necessarily satisfy either the differential equations or the boundary conditions. This is gradually warped into a solution of the problem by the algorithm. The method thus resembles quasi-linearization, but it is distinct. The major differences are in the method of imbedding solutions and in the use of the continuation method (a differential equation analog of Newton's method) to reduce the sensitivity of convergence. The method reduces to quasi-linearization when Euler's method is used as the integration algorithm. Two numerical examples are presented.

I. Introduction

IT is well-known that the necessary conditions of optimal control theory[1] or of the calculus of variations[2] can be used to reduce dynamic optimization problems to two point boundary value problems (TPBVPs). In turn, a TPBVP can be reduced to a nonlinear operator equation on an appropriate Banach space. This reformulation can be accomplished in at least two distinct ways: by initial value techniques[3] or by integral equation techniques.[4]

The solution of the resulting nonlinear equations is a fundamental problem of applied mathematics to which a large amount of research has been directed. By far the most well-known technique used is Newton's method as generalized to Banach spaces by Kantorovich.[5] When Newton's method is applied to the initial value problem, the resulting algorithm is called a shooting method; when it is applied to the integral equation formulation, quasi-linearization results.

Although Newton's method is the most popular it is not the only method for the solution of nonlinear equations. In particular, the *continuation method*[6-8] has been developed in an attempt to extend the region of convergence obtained from Newton's method. The idea is to replace the iterative Newton's method by a differential equation which is obtained as the result of a certain imbedding process.

Recently, a number of researchers[3,8-13] have applied the continuation method to TPBVPs. With the exception of Bosarge,[8] all of this work has been based on the initial value approach. The differences arise from the particular imbedding scheme used.

The present paper begins with a discussion of the continuation method, with emphasis on its numerical application. Then the continuation method is combined with a new imbedding technique to derive an algorithm that bears a strong resemblance to quasi-linearization. The paper concludes with two numerical examples and an example that illustrates an increase in the radius of convergence for the new method when compared with quasi-linearization. It is demonstrated in the Appendix that the method reduces to quasi-linearization when Euler's method is used as the integration algorithm.

Presented as Paper 73-890 at the AIAA Guidance and Control Conference, Key Biscayne, Fla., August 20–22, 1973; submitted November 21, 1973; revision received March 18, 1974.

Index categories: Aircraft Flight Operations; VTOL Flight Operations; Marine Vessel Trajectories, Stability, and Control.

* Project Engineer, Systems and Research Center. Member AIAA.
† Ph.D. Candidate, Department of Electrical Engineering.

II. Continuation Method for Nonlinear Equations

Theoretical Development

Suppose that a function $F: R^n \times R \rightarrow R^n$ is given, and that the roots of the equation

$$F(x, b) = 0 \tag{1}$$

are to be found for $b_0 \leqq b \leqq b_1$. Suppose further that a starting point x_0 is given such that

$$F(x_0, b_0) = 0 \tag{2}$$

that F is continuously differentiable in an open neighborhood $E \subset R^n \times R$ of (x_0, b_0), and that $F_x(x_0, b_0)$ is invertible, where F_x is the partial derivative of F with respect to x. Then the implicit function theorem[14] insures the existence of a unique function $x(b)$ such that

$$F[x(b), b] = 0 \tag{3}$$

in a neighborhood $W \subset R$ of b_0. Furthermore, the function $x(b)$ is differentiable in W, and its derivative is

$$dx/db = -F_x^{-1}[x(b), b]F_b[x(b), b] \tag{4}$$

where F_b is the partial derivative of F with respect to b. The idea of the continuation method is to integrate Eq. (4), the *continuation equation*, from b_0 to b_1 with initial condition $x(b_0) = x_0$.

One application of the continuation method is that of finding a root $\tilde{x}$ of the equation

$$f(x) = 0 \tag{5}$$

where $f: R^n \rightarrow R^n$ is continuously differentiable in a neighborhood of $\tilde{x}$. A possible function F for this case is

$$F(x, b) = f(x) - (1-b)f(x_0) \tag{6}$$

where x_0 is an initial guess at $\tilde{x}$. The continuation equation for this F is

$$dx/db = -f_x^{-1}[x(b)]f(x_0) \tag{7}$$

If Eq. (7) can be integrated from $x(0) = x_0$ to $x(1)$, then $x(1)$ is a root of Eq. (5).

Other choices of F than that given in Eq. (6) may be useful. For example, the function

$$F(x, b) = f(x) - (1-b)^2 f(x_0) \tag{8}$$

was found to give a more uniform parameterization of x(b) than Eq. (6) for one problem tested.

Existence of a solution to Eq. (7) is assured if $f_x(\tilde{x})$ is invertible and x_0 is sufficiently close to $\tilde{x}$. The inverse function theorem[15] implies the existence of an inverse to f in a neighborhood of $\tilde{x}$. Therefore, the function $x(b)$ can be taken as

$$x(b) = f^{-1}[(1-b)f(x_0)] \tag{9}$$

This result plays the role of a local convergence theorem for the continuation method. Note that the essential hypothesis [$f_x(\bar{x})$ invertible] is the same as that required for the Kantorovich local convergence theorem for Newton's method.[6] More elaborate convergence theorems can be proved using results from the theory of differential equations.[7,8]

It is instructive to compare the continuation method for solution of Eq. (5) with Newton's method. If Eq. (7) is integrated numerically by using Euler's method[16] with a step size $\Delta b = 1$, the result is

$$x_1 = x(0) - f_x^{-1}[x(0)]f[x(0)] \tag{10}$$

where x_1 is the approximation to $x(1)$. But this is just a single Newton iteration. Clearly more sophisticated integration techniques can be used to provide a family of iterative techniques similar to Newton's method.

Numerical Considerations

To apply the continuation method to a particular problem, it is necessary to integrate the differential equation (4). The use of any numerical integration scheme implies the buildup of roundoff and truncation error in the computed solution. However, numerical integration of the continuation equation is simplified by Eq. (3), which provides n independent first integrals of the solution. These first integrals can be used in conjunction with Newton's method to correct numerical errors in the solution. Thus, if the approximation x_k to $x(b_k)$ has been computed and $F(x_k, b_k)$ is not within some specified tolerance of zero, Newton's method can be applied to the problem

$$F(y, b_k) = 0 \qquad (b_k \text{ fixed}) \tag{11}$$

using x_k as a starting point. Roundoff and truncation error can thus be controlled. An integration algorithm with this feature is easily developed.[3]

III. Application of the Continuation Method

Application to TPBVPs

Consider the TPBVP

$$\dot{y} = f(y, t) \tag{12}$$

$$\psi_1[y(t_1)] = 0 \tag{13}$$

$$\psi_2[y(t_2)] = 0 \tag{14}$$

where y is an n-vector, ψ_1 is a p-vector function of y, and ψ_2 is an $(n-p)$-vector function of y. Appropriate technical assumptions are made to assure the existence of solutions to Eq. (12).[15] Problems with variable initial and terminal times can be reduced to the abovementioned form by a well-known transformation.[17]

Choose differentiable starting functions $y_s(t)$, $t_1 \leqq t \leqq t_2$, and imbed the original problem into the problem

$$\dot{y} = f(y, t) + (1-b)[\dot{y}_s - f(y_s, t)] \tag{15}$$

$$\psi_1[y(t_1)] - (1-b)\psi_1[y_s(t_1)] = 0 \tag{16}$$

$$\psi_2[y(t_2)] - (1-b)\psi_2[y_s(t_2)] = 0 \tag{17}$$

where $0 \leqq b \leqq 1$. Note that for $b = 0$ the solution is $y_s(t)$, and for $b = 1$, Eqs. (15–17) reduce to Eqs. (12–14).

For any initial condition y_1, the solution of Eq. (15) has the form $y(t; y_1, b)$. Therefore, the problem reduces to solving the n equations

$$\psi_1(y_1) - (1-b)\psi_1[y_s(t_1)] = 0 \tag{18}$$

$$\psi_2[y(t_2; y_1, b)] - (1-b)\psi_2[y_s(t_2)] = 0 \tag{19}$$

for the n initial conditions y_1. This problem is clearly equivalent to Eq. (1) when the identification

$$F(y_1, b) = \begin{bmatrix} \psi_1(y_1) - (1-b)\psi_1[y_s(t_1)] \\ \psi_2[y(t_2; y_1, b) - (1-b)\psi_2[y_s(t_2)]] \end{bmatrix} \tag{20}$$

is made. Notice that in this case evaluation of F requires integration of Eq. (15).

To apply the continuation method, the derivatives F_{y_1} and F_b must be computed. These are

$$F_{y_1}(y_1, b) = \begin{bmatrix} \psi_{1y}(y_1) \\ \psi_{2y}[y(t_2; y_1, b)] \dfrac{\partial y(t_2; y_1, b)}{\partial y_1} \end{bmatrix} \tag{21}$$

$$F_b(y_1, b) = \begin{bmatrix} \psi_1[y_s(t_1)] \\ \psi_{2y}[y(t_2; y_1, b)] \dfrac{\partial y(t_2; y_1, b)}{\partial b} + \psi_2[y_s(t_2)] \end{bmatrix} \tag{22}$$

where ψ_{1y} and ψ_{2y} are the $p \times n$ and $(n-p) \times n$ partial derivative matrices of ψ_1 and ψ_2 with respect to y. The partial derivatives $\partial y/\partial y_1$, $\partial y/\partial b$ are obtained as the solutions of the linear differential equations[15]

$$(d/dt)(\partial y/\partial y_1) = f_y[y(t; y_1, b), t](\partial y/\partial y_1) \tag{23}$$

$$(d/dt)(\partial y/\partial b) = f_y[y(t; y_1, b), t](\partial y/\partial b) - \{\dot{y}_s(t) - f[y_s(t), t]\} \tag{24}$$

The initial conditions are

$$\partial y/\partial y_1|_{t=t_1} = I_n \tag{25}$$

$$\partial y/\partial b|_{t=t_1} = 0 \tag{26}$$

where I_n is the $n \times n$ identity matrix and 0 is the 0 n-vector. The continuation equation for the problem is

$$dy_1/db = -F_{y_1}^{-1}[y_1(b), b]F_b[y_1(b), b] \tag{27}$$

with the initial condition $y_1(0) = y_s(t_1)$. A comparison of this method with quasi-linearization is given in the Appendix.

Numerical Considerations

To apply a numerical integration routine to Eq. (27), the derivative dy_1/db must be evaluated for a number of values of b. This is accomplished by simultaneously integrating Eqs. (15, 23, and 24) so that $F_{y_1}[y_1(b), b]$ and $F_b[y_1(b), b]$ can be evaluated from Eqs. (21) and (22). Then dy_1/db is obtained as the solution of the linear system

$$F_{y_1}[y_1(b), b]\, dy_1/db = -F_b[y_1(b), b] \tag{28}$$

Thus each evaluation of the right-hand side of Eq. (27) requires integration of n nonlinear and $n^2 + n$ linear differential equations, and solution of a system of n linear equations. Therefore, an integration routine minimizing the number of right-hand side evaluations per integration step is desirable. Our experience with a satisfactory integration algorithm[3] on an example problem is presented in Sec. IV to aid the reader in evaluating the computational efficiency of the method.

Application to Trajectory Optimization

Consider the optimal control problem

$$\underset{u(\cdot)}{\text{minimize}}\ \phi[x(t_2)] + \int_{t_1}^{t_2} f_0(x, u)\, dt \tag{29}$$

$$\dot{x} = f(x, u) \tag{30}$$

$$x(t_0) = x_0, \qquad \psi[x(t_2)] = 0 \tag{31}$$

Problems in which t appears in the differential equations and integrand are easily converted to the form of Eqs. (29) and (30) by a well-known transformation.[3]

We assume that the optimization problem is nonsingular and contains no state variable and/or control variable inequality constraints. Although extensions of the method to such problems may be possible, none is considered in this paper.

The Hamiltonian for the problem is

$$H[x(t), u(t), p(t)] = f_0(x, u) + p^T(t)f(x, u) \tag{32}$$

The necessary conditions for the optimality of a control $u(\cdot)$ and a state trajectory $x(\cdot)$ are the existence of a costate trajectory $p(\cdot)$ such that

$$H_u[x(t), u(t), p(t)] = 0 \tag{33}$$

$$\dot{x}(t) = H_p^T[x(t), u(t)] \tag{34}$$

$$\dot{p}(t) = -H_x^T[x(t), u(t), p(t)] \tag{35}$$

$$x(t_1) = x_1 \tag{36}$$

$$\psi[x(t_2)] = 0 \tag{37}$$

$$\phi_x^T[x(t_2)] + \psi_x^T[x(t_2)]\nu = p(t_2) \tag{38}$$

where ν is a multiplier vector with the same dimension as ψ.

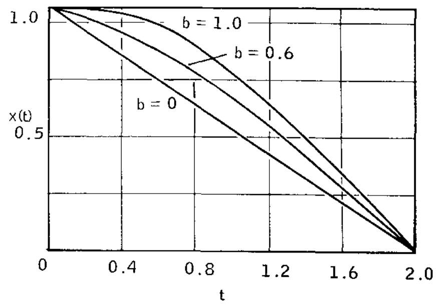

Fig. 1 Convergence of $x(t)$ from the starting function ($b = 0$) to the TPBVP solution ($b = 1$).

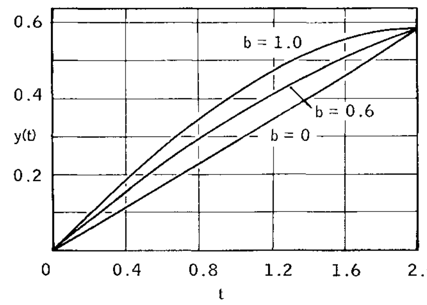

Fig. 2 Convergence of $y(t)$ from the starting function ($b = 0$) to the TPBVP solution ($b = 1$).

Assuming that $H_{uu} > 0$ (nonsingularity) and that $\psi_x[x(t_2)]$ has full rank, Eqs. (33–38) can be reduced to a TPBVP of the type considered previously. One way of doing this is to solve Eq. (33) for $u[x(t), p(t)]$ and to rewrite Eqs. (34) and (35) as

$$\dot{x}(t) = H_p{}^T\{x(t), u[x(t), p(t)]\} \tag{39}$$

$$\dot{p}(t) = -H_x{}^T\{x(t), u[x(t), p(t)], p(t)\} \tag{40}$$

Then Eqs. (39) and (40) play the role of Eq. (12), and Eqs. (36–38) replace Eqs. (13) and (14). Notice that this approach requires that starting functions $x_s(t)$, $p_s(t)$ be chosen for the state and costate.

A variation of this idea is to choose a starting control function $u_s(t)$ as well as $x_s(t)$ and $p_s(t)$, and to replace Eq. (33) by

$$H_u[x(t), u(t), p(t)] - (1-b)H_u[x_s(t), u_s(t), p_s(t)] = 0 \tag{41}$$

This is solved for $u[x(t), p(t), b, t]$ and inserted in Eqs. (39) and (40) above. In this case the differential equations corresponding to Eq. (12) also depend on the parameter b, so a slight modification of Eq. (24) is required. [We add the explicit derivative of f with respect to b to Eq. (24).]

Another variation avoids the necessity of guessing starting costate functions. One chooses $x_s(t)$, $u_s(t)$ and integrates

$$\dot{p}_s(t) = -H_x{}^T[u_s(t), x_s(t), p_s(t)] \tag{42}$$

to obtain $p_s(t)$ for some choice of initial condition on p_s. This is illustrated in the second example below.

A final variation maintains the advantage of choosing $u_s(t)$ without actually specifying it. Given $x_s(t)$, one lets $u_s(t)$ be the u that minimizes the quadratic function

$$J(u) = \{\dot{x}_s(t) - H_p{}^T[x_s(t), u]\}^T Q\{\dot{x}_s(t) - H_p{}^T[x_s(t), u]\} \tag{43}$$

This idea is also illustrated in the second example below.

Additional Considerations

The advantage of selecting a starting trajectory $x_s(t)$ is that advantage can be taken of any a priori known characteristics of the optimal path (e.g., a good approximation of the optimal path is available). The path $x_s(t)$ does not have to satisfy any differential equations. It may be possible to specify a good approximation of the controller $u_s(t)$ that goes with $x_s(t)$. It is generally difficult to specify good starting costate functions $p_s(t)$.

The minimum path information required by this optimization procedure is $x_s(t)$. The controller $u_s(t)$ can be obtained by minimizing Eq. (43) at each t. The starting costate can then be obtained by integrating Eq. (42). Equation (41) is used to convert to the TPBVP. This approach is illustrated below.

IV. Examples

Orbital Intercept Problem

To illustrate the application of the continuation method to the solution of two point boundary value problems, an orbital intercept problem due to Kenneth and McGill[18] was solved. The problem is that of transferring a space vehicle from a specified position 300 miles above the Earth to a second position 600 miles above the Earth in a fixed transit time. The flight is free-fall, so that the solution is obtained by proper choice of the initial velocity vector.

The equations of motion (neglecting the perturbations of the moon) are

$$\ddot{x} = -Kx/r^3 \tag{44}$$

$$\ddot{y} = -Ky/r^3 \tag{45}$$

$$\ddot{z} = -Kz/r^3 \tag{46}$$

with boundary conditions

$$x(0) = 1.076 \qquad x(2) = 0 \tag{47}$$

$$y(0) = 0 \qquad y(2) = 0.576 \tag{48}$$

$$z(0) = 0 \qquad z(2) = 0.997661 \tag{49}$$

The unit of length is the radius of the Earth and the gravitational constant K was normalized to one. This resulted in a time unit of 800.43 sec.

The starting functions were taken to be the straight lines with end points given by Eqs. (47–49). The method gradually warps these starting functions into solutions of the two point boundary value problem. Plots of $x(t)$, $y(t)$, and $z(t)$ are shown in Figs. 1–3 for $b = 0$ (starting trajectory), $b = 0.6$ (intermediate), and $b = 1$ (solution path).

Convergence was obtained for several integration techniques. Convergence without any Newton's method corrections was obtained by Runge-Kutta and Adams integrating schemes using 10 integration steps (i.e., $\Delta b = 0.1$). Convergence was also obtained using one Runge-Kutta step with one Newton-Raphson correction. Since one Runge-Kutta step requires four integrations of the differential equations this result is roughtly equivalent to Kenneth and McGill's result which required three integrations of the differential equations.

Continuous Stirred-Tank Reactor (CSTR)

This example illustrates the application of the continuation method to the trajectory optimization problem. The CSTR problem, as stated by Jamshidi[12] is

$$\text{minimize} \int_0^{0.78} (x_1{}^2 + x_2{}^2 + 0.1u^2)\, dt \tag{50}$$

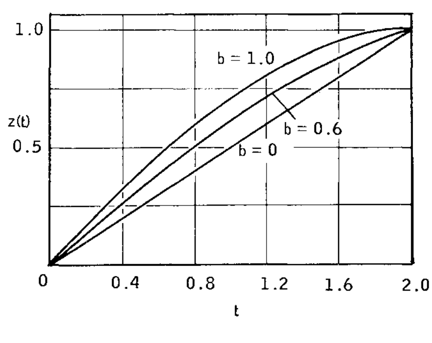

Fig. 3 Convergence of $z(t)$ from the starting function ($b = 0$) to the TPBVP solution ($b = 1$).

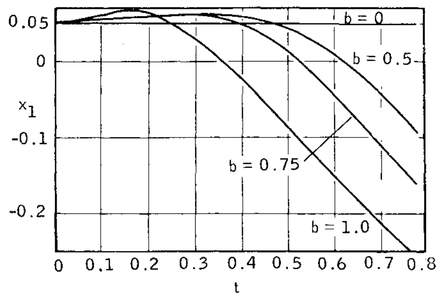

Fig. 4 Convergence of $x_1(t)$ from the starting function ($b = 0$) to the optimal solution ($b = 1$).

$$\dot{x}_1 = -(1+2x_1)+(x_2+0.5)\exp[25x_1/(x_2+2)]-(x_1+0.25)u \quad (51)$$

$$\dot{x}_2 = -x_2-(x_2+0.5)\exp[25x_1/(x_2+2)]+0.5 \quad (52)$$

$$x_1(0) = 0.05, \qquad x_2(0) = 0 \quad (53)$$

Initial functions

We assume that differentiable starting functions $x_{1s}(t)$, $x_{2s}(t)$ are specified that satisfy Eq. (53). Then Eq. (51) can be solved to obtain

$$u_s(t) = [x_1+0.25]^{-1}\{-\dot{x}_{1s}(t)-[1+2x_{1s}(t)]+ [x_{2s}(t)+0.5]\exp[25x_1/(x_2+2)]\} \quad (54)$$

The necessary conditions, Eq. (38), require that

$$p_1(0.78) = p_2(0.78) = 0 \quad (55)$$

Then a starting multiplier vector $p_s(t)$ satisfying these conditions can be obtained by integrating

$$\dot{p}_s(t) = -(\partial H/\partial x)[x_s(t), u_s(t), p_s(t)]^T \quad (56)$$

using Eq. (55) as terminal condition.

Continuation method equation

We show functionally how the continuation method is applied, and leave the algebraic details to the reader. Application of Eq. (41) gives

$$u[x_1(t), p_1(t), b, t] = 5\{[x_1(t)+0.25]p_1(t)+ (1-b)[\![0.2u_s(t)-[x_{1s}(t)+0.25]p_{1s}(t)]\!]\} \quad (57)$$

Then Eqs. (51) and (52) become, in terms of the continuation equations (15),

$$\dot{x}_1(t) = f_1\{x_1(t), x_2(t), u[x_1(t), p_1(t), b, t]\} \quad (58)$$

$$\dot{x}_2(t) = f_2[x_1(t), x_2(t)]+(1-b)\{\dot{x}_{2s}(t)-f_2[x_{1s}(t), x_{2s}(t)]\} \quad (59)$$

The costate vector equation converts to

$$\dot{p}(t) = -(\partial H/\partial x)\{x(t), p(t), u[x_1(t), p_1(t), b, t]\}^T \quad (60)$$

The solutions of Eqs. (58–60) are functions of t, b, and the initial conditions on the state and costate vectors at $t = 0$. Since $x_{1s}(0)$ and $x_{2s}(0)$ satisfy Eq. (53), Eq. (18) demands that the initial state vector be fixed at (0.05, 0). Since $p_s(t = 0.78) = 0$, Eq. (19) reads, in terms of the remaining variables

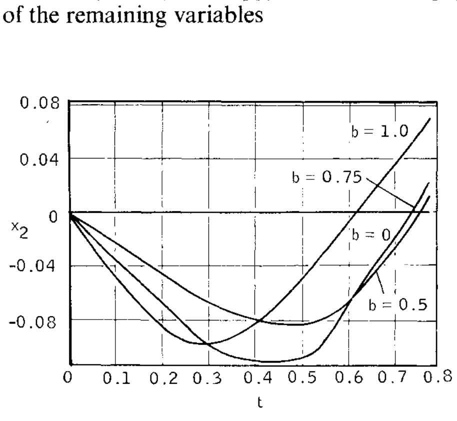

Fig. 5 Convergence of $x_2(t)$ from the starting function ($b = 0$) to the optimal solution ($b = 1$).

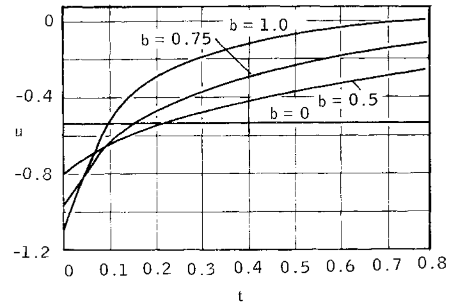

Fig. 6 Convergence of $u(t)$ from its initial value $u_s(t)$ to the optimal value ($b = 1$).

$$p_1(0.78, p_{10}, p_{20}, b) = 0 \quad (61)$$

$$p_2(0.78, p_{10}, p_{20}, b) = 0 \quad (62)$$

where p_{10} and p_{20} are the initial values of the costate vector components. In summary, Eqs. (58–60) correspond to Eq. (15) and Eqs. (61) and (62) correspond to Eqs. (18) and (19) in the application of the continuation method to TPBVP's.

Numerical results

Initial functions were chosen as

$$x_{1s}(t) = 0.05, \; x_{2s}(t) = 0, \qquad 0 \leqq t \leqq 0.78 \quad (63)$$

The continuation equations were integrated using the Adams predictor, Newton's method corrector (when needed) algorithm, with a step-size of $\Delta b = 0.025$. Once started, the Adams formula requires only one evaluation of the right-hand side of Eq. (27) per integration step. Newton corrections were required on only 5 steps. Because of the Runge-Kutta startup, a total of 55 inner loop integrations were performed. The convergence of the solution from the initial guess ($b = 0$) to the solution of the optimization problem ($b = 1$) is illustrated in Figs. 4–6 in terms of $x_1(t)$, $x_2(t)$, and $u(t)$.

Two comments on the solution are in order. First, sufficiency calculations verify that it is a local minimizing solution. Second, Jamshidi apparently solved the problem as stated in his Ref. 13 instead of as stated in Eqs. (50–53).

Analytical Example

This example illustrates that an extended region of convergence can be achieved by the methods proposed herein as compared to quasi-linearization. The problem is to find a function $x(t)$ satisfying

$$\dot{x}(t) = 0 \qquad 0 \leqq t \leqq 1 \quad (64)$$

$$x(1)\exp(-|x(1)|) = 0 \quad (65)$$

This is an extremely simple TPBVP for which quasi-linearization and the method of this paper can be compared analytically.

It is easily verified that Eqs. (64) and (65) are equivalent to the problem of finding x such

$$x\exp(-|x|) = 0 \quad (66)$$

Moreover, the continuation method for Eqs. (64) and (65) as outlined in Sec. III is equivalent to the continuation method for Eq. (66) as given in Sec. II, and quasi-linearization for Eqs. (64) and (65) is equivalent to Newton's method for Eq. (66).

The continuation equation for Eq. (66) with the initial guess x_0 is

$$\frac{dx}{db} = \begin{cases} [-x_0/(1-x)]\, e^{(x-x_0)} & x \geqq 0 \\ [-x_0/(1+x)]\, e^{-(x-x_0)} & x < 0 \end{cases} \quad (67)$$

For $x_0 \in (-1, 1)$ the solution $x(b)$ moves smoothly from $x(0) = x_0$ to $x(1) = 0$, so that the continuation method converges. By comparison, the region of convergence of Newton's method is $(-\frac{1}{2}, \frac{1}{2})$, as suggested by Fig. 7.

Although this example is somewhat contrived, it does illustrate the difference in philosophy between Newton's method and the continuation method. Newton's method assumes that the solution can be found in one step using a first-order expansion

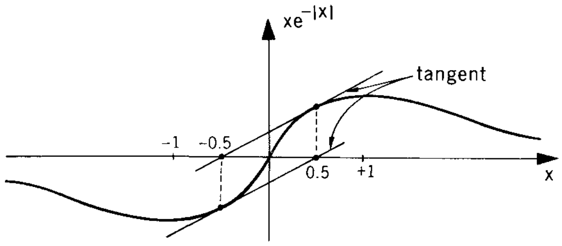

Fig. 7 Nonconvergence of Newton's method.

about the current guess. Convergence is quadratic if that point is within the radius of convergence. However, the method may diverge for guesses close (but not close enough) to the solution. The continuation method is a smaller step method used with the expectation of surer convergence. It may take more computation in arriving at the answer than Newton's method. In fact, the crudest method for solving the problem via the continuation method is to apply Newton's method to Eq. (6) N times. The first time $b = 1/N$ and the initial guess is x_0. The second time $b = 2/N$ and the initial guess is the solution $x(1/N)$ of the first application. The last time $b = 1$ and the initial guess is $x(N-1/N)$ of the $(N-1)$th application. More sophisticated integration algorithms, of course, reduce the total amount of computations required to obtain the solution.

V. Conclusions

A new algorithm for solving two-point boundary value problems is presented and is based upon two ideas. The first is a new method for imbedding solutions of differential equations as functions of a parameter. This allows the initial solution to be any (reasonable) differentiable function of time not necessarily satisfying the original set of differential equations. The final solution will satisfy them. The second idea is the continuation method. This provides a practical procedure for obtaining the solutions of the differential equations as functions of the parameter, and for solving the TPBVP.

A good application of this new algorithm is that of finding solutions for trajectory optimization problems. The method is illustrated for nonsingular problems with no inequality constraints. The minimal initial data required is a differentiable sketch of the state vector as a function of time. A procedure for obtaining an initial controller to go with the initial state function is described. A method for calculating a costate vector function for these initial functions is demonstrated for a particular terminal surface description. This is important because it is, in general, difficult to prespecify appropriate starting costate functions for such problems. The last idea can be generalized for other terminal surface descriptions.

Appendix: Comparison of the Method with Quasi-linearization

As noted in Sec. II, integrating the continuation equation with a one-step Euler method results in a single iteration of Newton's method. This fact allows a very explicit comparison between quasi-linearization and the method outlined previously.

Applying Euler's method to Eq. (27) with $\Delta b = 1$ gives

$$y_1(1) = y_1(0) - F_{y_1}{}^{-1}[y_1(0), 0]F_b[y_1(0), 0] \tag{68}$$

which is equivalent to

$$F_{y_1}[y_1(0), 0]\,[y_1(0) - y_1(1)] = F_b[y_1(0), 0] \tag{69}$$

Substituting from Eq. (21), Eq. (22) gives

$$\psi_{1y}[y_1(0)]\,[y_1(0) - y_1(1)] = \psi_1[y_s(t_1)] \tag{70}$$

$$\psi_{2y}\{y[t_2\,; y_1(0), 0]\}\,\frac{\partial y[t_2\,; y_1(0), 0]}{\partial y_1}[y_1(0) - y_1(1)] = \psi_{2y}\{y[t_2\,; y_1(0), 0]\}\,\frac{\partial y[t_2\,; y_1(0), 0]}{\partial b} + \psi_2[y_s(t_2)] \tag{71}$$

Since $y_1(0) = y_s(t_1)$ and $y[t_2\,; y_1(0), 0] = y_s(t_2)$, rearranging terms gives

$$\psi_1[y_s(t_1)] + \psi_{1y}\{y_s(t_1)[y_1(1) - y_s(t_1)]\} = 0 \tag{72}$$

$$\psi_2[y_s(t_2)] + \psi_{2y}[y_s(t_2)]\left\{\frac{\partial y[t_2\,; y_s(t_1), 0]}{\partial y_1}[y_1(1) - y_s(t_1)] + \frac{\partial y[t_2\,; y_s(t_1), 0]}{\partial b}\right\} = 0 \tag{73}$$

But from Eqs. (23) and (24) it follows that

$$\frac{\partial y[t_2\,; y_s(t_1), 0]}{\partial y_1}[y_1(1) - y_s(t_1)] + \frac{\partial y[t_2\,; y_s(t_1), 0]}{\partial b} = \bar{y}(t_2) - y_s(t_2) \tag{74}$$

where $\bar{y}(t)$ is the solution of

$$(d/dt)[\bar{y}(t) - y_s(t)] = f_y[y_s(t), t][\bar{y}(t) - y_s(t)] - [\dot{y}_s(t) - f_{y_s}(t)] \tag{75}$$

with initial condition $y(t_0) = y_1(1)$. Canceling $\dot{y}_s(t)$ from both sides of Eq. (75) gives

$$(d/dt)y(t) = f[y_s(t), t] + f_y[y_s(t), t][y(t) - y_s(t)] \tag{76}$$

and substituting Eq. (74) in Eq. (73) gives

$$\psi_2[y_s(t_2)] + \psi_{2y}[y_s(t_2)]\,[\bar{y}(t_2) - y_s(t_2)] \tag{77}$$

Equations (70, 77, and 76) represent the linearized TPBVP of quasi-linearization.[17]

For less trivial integration methods, the techniques are quite different. Note, for example, that in quasi-linearization only linear equations are integrated, while in the continuation method with the imbedding described above, the nonlinear equation (15) is integrated each time.

References

[1] Athans, M. and Falb, P. L., *Optimal Control*, McGraw-Hill, New York, 1966.

[2] Bliss, G. A., *Lectures on the Calculus of Variations*, Univ. of Chicago Press, Chicago, 1946.

[3] Scharmack, D. K., "An Initial Value Method for Trajectory Optimization Problems," *Advances in Control Systems*, 1st ed., Vol. 5, Academic Press, New York, 1967, pp. 51–132.

[4] Falb, P. L. and deJong, J. L., *Some Successive Approximation Methods in Control and Oscillation Theory*, Academic Press, New York, 1969.

[5] Kantorovich, L. V. and Akilov, G. P., *Functional Analysis in Normed Spaces*, Macmillan, New York, 1966.

[6] Ortega, J. M. and Rheinbold, W. C., *Iterative Solutions of Nonlinear Equations*, Academic Press, New York, 1970.

[7] Yakovlev, M. N., "Solution of Nonlinear Equations by the Method of Differentiation with Respect to a Parameter," TT F-254, Sept. 1965, NASA.

[8] Bosarge, W. E., Jr., "Infinite Dimensional Iterative Methods," Ph.D. Thesis, 1969, Brown University, Providence, R.I.

[9] Brown, K. R., Jr. and Johnson, G. W., "Rapid Computation of Optimal Trajectories," *IBM Journal of Research and Development*, Vol. 11, No. 4, July 1967, pp. 373–382.

[10] Glasser, D., "Numerical Solution of Two-Point Boundary Value Problems on Total Differential Equations," *SIAM Journal of Numerical Analysis*, Vol. 6, No. 4, Dec. 1969, pp. 591–597.

[11] Hontoir, Y. and Cruz, J. B., Jr., "A Manifold Imbedding Algorithm for Optimization Problems," *Automatica*, Vol. 8, No. 5, Sept. 1972, pp. 581–587.

[12] Jamshidi, M., "On the Imbedding Solution of a Class of Optimal Control Problems," *Automatica*, Vol. 8, No. 5, Sept. 1972, pp. 637–640.

[13] Roberts, S. M. and Shipman, J. S., *Two Point Boundary Value Problems: Shooting Methods*, Elsevier, New York, 1972.

[14] Rudin, W., *Principles of Mathematical Analysis*, McGraw-Hill, New York, 1964.

[15] Coddington, E. A. and Levinson, N., *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.

[16] Hildebrand, F. B., *Introduction to Numerical Analysis*, McGraw-Hill, New York, 1946.

[17] Bryson, A. E. and Yo, Y. C., *Applied Optimal Control*, Blaisdel, Waltham, Mass., 1969.

[18] McGill, R. and Kenneth, P., "Solution of Variational Problems by Means of a Generalized Newton-Raphson Operator," *AIAA Journal*, Vol. 2, No. 10, Oct. 1964, pp. 1761–1766.